# THE RELATIONS OF ANALYSIS AND MATHEMATICAL PHYSICS.*

*AN ADDRESS BEFORE THE INTERNATIONAL CONGRESS OF MATHEMATICIANS, ZÜRICH, AUGUST 9, 1897.*

BY PROFESSOR HENRI POINCARÉ.

## I.

Doubtless you are often asked what is the utility of mathematics and whether its nicely constructed theories, drawn entirely from the mind, are not artificial products of our caprice.

Among the persons who ask this question, I must make a distinction. The practical class demand of us nothing but means of getting money. These do not deserve to be answered. Rather ought it to be demanded of them what is the good of accumulating so much wealth and whether, in order to have time for its acquisition, it is necessary to neglect art and science, which alone render the soul capable of enjoying it,

*Et, propter vitam, vivendi perdere causas.*

Moreover, a science produced with a view single to its applications is impossible; truths are fruitful only if they are concatenated; if we cleave to those only of which we expect an immediate result, the connecting links will be lacking, and there will be no longer a chain.

The men who are most disdainful of theory find therein, without suspecting it, a daily aliment. Were they deprived of this aliment, progress would quickly be arrested, and we should very soon settle into Chinese immobility.

But we have sufficiently occupied ourselves with the uncompromising practicians; besides these there are those who are curious about Nature only and who ask us if we are in position to help them to a better comprehension of her. In response we have only to show them two monuments, already rough-hewn, celestial mechanics and mathematical physics. They would doubtless concede that these monuments are well worth the labor they have cost. But this is not enough.

Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore, it has

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a philosophic end, and, I venture to say, an end esthetic. It ought to incite the philosopher to search into the notions of number, space, and time; and, above all, adepts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of numbers and of forms; they are amazed when a new discovery discloses for them an unlooked for perspective; and the joy they thus experience, has it not the esthetic character although the senses take no part in it? Only the privileged few are called to enjoy it fully, it is true; but is it not the same with all the noblest arts? Hence I do not hesitate to say that mathematics deserves to be cultivated for its own sake and that the theories not admitting of application to physics deserve to be studied as well as the others.

Even if the physical and esthetic ends were not conjoint, we ought to sacrifice neither the one nor the other. But these two ends are inseparable, and the best means of attaining the one is to aim at the other, or at least never to lose sight of it, a fact I shall now try to demonstrate by showing precisely the nature of the relations between pure science and its applications.

The mathematician ought not to be for the physicist a simple provider of formulæ; between the two there ought to be more intimate collaboration. Mathematical physics and pure analysis are not simply adjacent powers maintaining the relations of good neighborhood; they interpenetrate, and their spirit is the same. This we shall the better comprehend when I shall have shown what physics receives from mathematics and what mathematics, in return, borrows from physics.

## II.

The physicist cannot demand of the analyst a revelation of new truth; the analyst can at best aid the physicist in the presentation of truth.

The time is past when people sought to anticipate experience, or to construct the world completely upon certain premature hypotheses. Of all the theories in which they delighted naïvely only a century ago, there remains to-day nothing but ruins.

Now all laws are derived from experience, but, to enunciate them, a special language is needed; ordinary language is too poor; it is besides too vague to express relations so delicate, so complex, and so precise. Here, then, is a prime

reason why the physicist cannot dispense with mathematics; it alone furnishes him with an adequate language.

Neither is it a small matter that a language be fit: not to pass from the domain of physics, the unknown man who invented the word *chaleur* doomed many generations to error; heat has been treated as a substance simply because it was designated by a substantive, and it was believed to be indestructible. On the other hand, he who invented the word *électricité* has had the unmerited good fortune implicitly to endow physics with a new law, of the conservation of electricity, which, by a pure hazard, is found to be exact, at least up to the present time.

Nay, to pursue the comparison, writers who embellish language, who treat it as an object of art, at the same time make it a suppler instrument, fitter to render the finer shades of thought. We see, therefore, how the analyst, who pursues a purely esthetic end, contributes thus to create a language better suited to the needs of the physicist.

But this is not all; law issues from experience but it does not do so immediately. Experience is individual, the law derived therefrom is general; experience is only approximate, law is precise or at least pretends to be. Experience is produced under conditions always complex; the enunciation of the law eliminates these complications. This is what is termed "elimination of systematic errors."

In a word, in order to derive law from experience, it is necessary to generalize, a necessity that imposes itself upon the most circumspect observer. But how generalize? Every particular truth can evidently be generalized in an infinity of ways. Among the thousand paths that open before us, it is necessary to make a choice, at least a provisional one; in this choice what shall guide us?

Analogy alone. But how vague this word! Primitive man takes cognizance of only rude analogies, those that strike the senses, analogies of color and sound. He would never have attempted to connect, for example, light and radiant heat.

What has taught us to discern those genuine, profound analogies that the eye does not see and only the reason divines, if not the mathematical spirit disregarding matter in order to attach itself to pure form? It is this spirit that has directed us to name with the same name things differing in respect to matter only; to name with the same name, for example, the multiplication of quaternions and that of whole numbers.

If quaternions, of which I just spoke, had not been so

promptly utilized by English physicists, many persons would doubtless have regarded them as only an idle dream; nevertheless, by teaching us to connect things separated by appearances, they would have already rendered us the apter to penetrate the secrets of nature.

Such are the services the physicist may expect from analysis; but in order that this science may render them, it must be cultivated on the largest scale, free from immediate preoccupation with utility; the mathematician must have labored as an artist. What we demand of him is to aid us in seeing, in discerning our way in the labyrinth that presents itself to us. He that sees best is he that has risen to the highest level. Examples abound, I will confine myself to the most striking.

The first will show us how it suffices to change language in order to perceive generalizations that were not at first suspected. When the Newtonian law was substituted for that of Kepler, only the elliptic motion was known. Now, so far as this motion is concerned, the two laws differ in form only; we pass from the one to the other by a simple differentiation. And yet from the law of Newton may be deduced, by an immediate generalization, all the effects of perturbations and the whole of celestial mechanics. Never, on the other hand, had we kept to Kepler's statement, would we have regarded the orbits of the disturbed planets—those complicated curves whose equations have never been written—as the natural generalization of the ellipse. The progress of observation would have served only to produce belief in chaos.

The second example deserves equal consideration. When Maxwell began his labors, the then recognized laws of electrodynamics, accounted for all the known facts. It was not a new experience that invalidated them, but, viewing them under a new aspect, Maxwell saw that the equations became more symmetric on the addition of a term, and, on the other hand, that this term was too small to produce appreciable effects by the old methods. We know that the *a priori* views of Maxwell awaited an experimental confirmation twenty years; or, if you prefer, Maxwell was twenty years in advance of experience. How was this triumph achieved?

Maxwell was profoundly impregnated with the sense of mathematical symmetry; would the case have been the same, had not others, before him, pursued this symmetry for its own beauty?

Maxwell was accustomed "to think in vectors," but vectors are introduced into analysis by the theory of imagin-

aries. And those who invented imaginaries little suspected that these would be turned to account in the study of the real world, a fact sufficiently proved by the name they gave them.

Maxwell, in a word, was perhaps not a skilled analyst but this skill would have been for him a useless and cumbrous baggage. On the other hand, he had in the highest degree a fine sense of mathematical analogies. On that account he became a thorough mathematical physicist.

The example of Maxwell teaches us yet another thing. How should the equations of mathematical physics be treated? Ought we simply to deduce from them all the consequences, and regard them as intangible realities? Far from it; that which they ought especially to teach us is that we can and must transform them, for thus shall we derive something useful from them.

The third example goes to show us how we may perceive mathematical analogies among phenomena which are neither apparently nor really so related physically that the laws of one phenomenon aid us in divining those of the other. A single equation, that of Laplace, is encountered in the theories of Newtonian attraction, of the motion of liquids, of the electrical potential, of magnetism, of the propagation of heat and in many others besides. What of it? These theories seem like images traced the one upon the other. They are mutually illuminated by each appropriating the language of the others; ask the electricians if they do not felicitate themselves on having invented the phrase, "flux de force," suggested by hydrodynamics and the theory of heat.

Thus mathematical analogies not only enable us to surmise physical analogies but are still useful when these latter are wanting.

To summarize, the end of mathematical physics is not merely to facilitate the numerical calculation of certain constants or the integration of certain differential equations. It is more, it is above all to disclose to the physicist the concealed harmonies of things by furnishing him with a new point of view.

Of all parts of analysis, those are the highest and purest, so to speak, which will be most productive in the hands of such as know how to use them.

## III.

Let us now consider what analysis owes to physics.

We should quite ignore the history of science if we failed

to recall how the desire to know nature has constantly exerted upon the development of mathematics their happiest influence.

In the first place, the physicist propounds problems whose solution he awaits at our hands. By proposing them to us he has paid largely in advance for the service we should render by their eventual solution. If I may be permitted to pursue my comparison with the fine arts, the pure mathematician who should forget the existence of the external world would be like a painter who knew how to combine colors and forms harmoniously but who had no models. His creative power would be quickly perverted.

Possible combinations of numbers and symbols form an infinite multitude. How shall we choose from this multitude such as are worthy to detain our attention? Shall we be guided solely by our caprice? This caprice, which, moreover, would ere long wear itself out, would doubtless lead us far asunder, and we should very soon cease to understand each other.

But this is only a minor phase of the question. Physics will doubtless prevent us from going astray, but it will preserve us from a much more formidable danger: that of revolving constantly in the same circle. History proves that physics has not only constrained us to choose among the hosts of problems that present themselves; it has forced upon us those which we had otherwise never attempted. How varied soever the imagination of man, nature is yet a thousand times richer. To pursue her, we should take paths that have been neglected, and these will often conduct to summits whence new landscapes will be revealed.

What more useful! It is with mathematical symbols as with physical realities; by comparing the different aspects of things we shall be able to comprehend their intimate harmony, which alone is beautiful and therefore worthy of our efforts.

The first example I will cite is so old that one is prone to forget it; it is, notwithstanding, the most important of all. The sole natural object of mathematical thought is the whole number. It is the external world that has imposed upon us the continuum, an invention doubtless of our own, but one that the external world forced us to make. Without it, there could be no infinitesimal analysis; all mathematical science would reduce to arithmetic or to the theory of substitutions.

On the other hand, we have devoted to the study of the continuum almost all our time and powers. Who would

regret it? Who would pretend that this time and energy have been lost?

Analysis unfolds before us infinite perspectives of which arithmetic has no suspicion; it shows us at a glance a grand assemblage, of simple and symmetric order. In number theory, on the contrary, where the unexpected reigns, the view is, so to speak, arrested at every step.

Doubtless it will be said that, apart from the whole number, there is no rigor and consequently no mathematical truth; that everywhere the whole number is concealed, and that we should try to render the veils transparent, even at the expense of interminable repetition. Let us not be such purists but let us be grateful to the *continuum*, which, if everything proceeds from the whole number, was alone capable of causing so great issue therefrom.

Need I, moreover, recall the fact that Mr. Hermite has derived a striking advantage from the introduction of continuous variables into the theory of numbers? Thus, the domain itself of the whole number has been invaded, and as a result order has been established there where disorder prevailed. Such is our debt to the *continuum* and, by consequence, to physical nature.

The series of Fourier is a precious instrument continually employed by the analyst; but Fourier invented it to solve a physical problem; if his problem had not been naturally set, we should never have dared to render to *discontinuum* its rights; we should for yet a long time have regarded the continuous functions as the only genuine functions.

The notion of *function* has been thereby considerably extended and, at the hands of certain analyst logicians, has received an unforeseen development. These analysts have thus ventured into the regions of the purest abstraction and have departed as far as possible from the real world. It was a physical problem, however, that furnished the point of departure.

After the series of Fourier, other analogous series were introduced into the domain of analysis; they enter by the same gate; they were conceived in view of applications. It is sufficient to cite those that have as elements the sphere functions or the functions of Lamé.

The theory of partial differential equations of the secsecond order has had an analogous history: it was specially developed by and for physics. If the analysts had been left to their natural tendencies, the following is probably how they would have viewed these equations, and how they would have chosen the limiting conditions:

Consider, for example, an equation between two variables $x$ and $y$ and a function $F$ of these two variables. They would have assumed $F$ and $\frac{dF}{dx}$ for $x = 0$. This was done, for example, by Mme. de Kowalewski in her celebrated memoir. But there is a host of other ways of putting the problem. $F$ may be given along an entire closed contour, as in the problem of Dirichlet, or the ratio of $F$ to $\frac{dF}{dn}$ may be given as in the theory of heat.

It is to physics that we owe all these ways of putting the problem. We may, then, say that without physics the theory of partial differential equations would not be known.

It is useless to multiply examples; I have said enough to warrant a conclusion. When the physicists require of us the solution of a problem, they do not thus impose drudgery upon us; on the contrary, we are under obligation to them.

## IV.

But this is not all; physics does not merely furnish problems for solution; it aids us in finding means therefor; and that in two ways: it causes us to surmise the solution, it suggests the proof.

I have mentioned above the fact that the equation of Laplace is met with in a host of far separated physical theories. We find it again in geometry in the theory of conformal representation, and in pure analysis in that of imaginaries. Thus in the study of functions of complex variables the analyst, besides the geometric image which is his usual instrument, finds several physical images that can be used with equal success. Thanks to these images he can see at a glance what pure deduction could show him only in succession. He thus collects the separate elements of the solution, and by a sort of intuition divines before he can demonstrate.

Divine before demonstrate! Need I recall the fact that all important discoveries are thus made? What truths of which the physical analogies give us a presentiment, and which we are not yet in position to establish by rigorous argument! For example, mathematical physics introduces a great number of developments in series. That these series converge, no one doubts; but mathematical certainty is wanting. These are so many assured conquests for the investigators who shall come after us.

Physics, on the other hand, does not merely furnish us with solutions; it also in certain measure provides arguments It will be sufficient to recall how Mr. Klein, in a question respecting Riemann surfaces, had recourse to the properties of electric currents. Arguments of this class, it is true, are not rigorous in the sense that the analyst attaches to this word.

And, in this connection, arises the question: How a demonstration that is not rigorous enough for the analyst, can be sufficient for the physicist? It seems there can not be two rigors, that rigor is or it is not, and that there where it is not, argument can not exist.

We will best comprehend this apparent paradox by recalling the conditions of the applicability of number to natural phenomena. Whence arise, in general, the difficulties encountered when one seeks to give a rigorous demonstration? One strikes them almost always when attempting to establish that such a quantity tends towards such a limit, or that such a function is continuous, or that it has a derivative. Now, the numbers that the physicist measures by experience are known to him only approximately, and, on the other hand, any function whatever always differs by as little as we please from a discontinuous function, and, at the same time, it differs as little as we please from a continuous function.

The physicist may accordingly suppose, at pleasure, that the function studied is continuous or that it is discontinuous; that it has a derivative or that it has not; and this, without fear of contradiction by either present or future experience. One understands how, with such liberty, he makes play of the difficulties that detain the analyst. He may always reason as if all the functions occurring in his calculations were entire polynomials.

Thus the view that suffices for physics is not such reasoning as analysis requires. It does not follow that the one is not able to aid the other.

So many physical observations have been already transformed into rigorous demonstrations that this transformation is easy to-day. Examples abound, if I did not fear, in citing them, to weary your attention, and if this conference had not been already too long.

I hope I have said enough to show that pure analysis and mathematical physics may be reciprocally helpful without either entailing sacrifice upon the other, and that each should rejoice in whatever exalts its associate.

# THE ROOTS OF POLYNOMIALS WHICH SATISFY CERTAIN LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

In volume 6 of the *Acta Mathematica*, Stieltjes has given a remarkable method for showing in how many different ways certain parameters in an important class of linear differential equations of the second order can be so determined that the equation shall have a polynomial solution, and in the course of the work the position of the roots of these polynomials is determined. It sometimes happens that the answer to the first part of the problem here referred to can be obtained more easily or more naturally by other methods. For instance in the case of the hypergeometric differential equation the forms in which solutions can be expanded in series show us at once in what cases we have a polynomial solution; and in the case of Lamé's equation the theorem of oscillation leads us most naturally (from some points of view) to the result.* There still remains the second part of the above problem, viz.: the determination of the position of the roots of the polynomials. The method of Stieltjes is connected with a problem in the equilibrium of particles on a straight line. By generalizing these considerations so as to bring in particles lying in a plane, we can, as I have shown,† obtain a theorem concerning the position of the roots of the polynomials, which, though in itself less far reaching (in some respects) than that of Stieltjes, gives us in the cases above referred to the information we want. I should like here to emphasize three points:

1. This method enables us to avoid the determination of an upper limit to the number of determinations of the parameters which give polynomial solutions.

2. It enables us to go beyond the cases considered by Stieltjes inasmuch as the singular points of the differential equation may now be complex.

3. Owing to the relatively small result we wish to attain

*Cf. my book: Ueber die Reihenentwickelungen der Potentialtheorie, Leipzig, Teubner, 1894. See pages 210–213.

†See pages 215–216 of the book just referred to.

it is possible to throw the proof into purely *algebraic* form; whereas Stieltjes's method involving, as it does, an existence-proof depends upon transcendental considerations.

I will now state the theorem and give its proof in the algebraic form to which I have referred, mentioning however, for the sake of brevity, only the case in which the singular points of the differential equation are real.

*Let* $\varphi(x) = (x - x_1)(x - x_2) \cdots (x - x_k)$ *be a polynomial which satisfies the differential equation:*

$$\frac{d^2y}{dx^2} + \left( \frac{a_1}{x - e_1} + \frac{a_2}{x - e_2} + \cdots + \frac{a_n}{x - e_n} \right) \frac{dy}{dx} + \frac{\psi(x)}{(x - e_1)(x - e_2) \cdots (x - e_n)} y = 0,$$

*in which* $\psi(x)$ *is a polynomial. Let* $e_1, e_2, \cdots, e_n$ *be real and unequal* $(e_1 < e_2 \cdots < e_n)$. *Then if* $a_1, a_2, \cdots, a_n$ *are real and greater than zero the roots* $x_1, x_2, \cdots, x_k$ *of* $\varphi(x)$ *must all be real and must lie in the interval* $e_1 \leqq x \leqq e_n$.

It is evident at once from the elements of the theory of multiple roots that $\varphi(x)$ can have no multiple root which is not equal to one of the quantities $e_1, e_2, \cdots, e_n$. Let $x_1$ be any root of $\varphi(x)$ different from $e_1, e_2, \cdots, e_n$. Then if we substitute $\varphi(x)$ in the differential equation and let $x = x_1$ we get:

$$\varphi''(x_1) + \left( \frac{a_1}{x_1 - e_1} + \frac{a_2}{x_1 - e_2} + \cdots + \frac{a_n}{x_1 - e_n} \right) \varphi'(x_1) = 0;$$

or dividing by $\varphi'(x_1)$ (which is not zero since $x_1$ is not a multiple root):

$$\frac{2}{x_1 - x_2} + \frac{2}{x_1 - x_3} + \cdots + \frac{2}{x_1 - x_k} + \frac{a_1}{x_1 - e_1} + \frac{a_2}{x_1 - e_2} + \cdots + \frac{a_n}{x_1 - e_n} = 0.*$$

Now if $\varphi(x)$ has complex roots with positive pure imaginary part let $x_1$ be that one (or one of those) whose pure imaginary part is greatest. Then the above equation involves a contradiction for the pure imaginary part of each term is negative or zero, and not all of them are zero since the $a$'s are not zero.

* The first member of this equation is a quantity (perhaps complex) whose conjugate gives both in magnitude and in direction the force acting upon the particle $x_1$ in the mechanical problem just referred to. The equation therefore gives the condition of equilibrium. The mechanical meaning of the following proof is obvious.

In a similar way we see that $\varphi(x)$ can have no complex root whose pure imaginary part is negative.

$x_1, x_2, \cdots, x_k$ are, therefore, all real. Suppose one of them were greater than $e_n$. Call this one (or, if there are more than one, the greatest of them) $x_1$. Then the above equation again involves a contradiction since no term is negative or zero.

In the same way we see that no root can be less than $e_1$.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

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## INFLEXIONAL LINES, TRIPLETS, AND TRIANGLES ASSOCIATED WITH THE PLANE CUBIC CURVE.

BY PROFESSOR HENRY S. WHITE.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

THE configuration of the nine inflexions of a nonsingular plane cubic and the twelve lines containing them three-and-three would seem too well known to merit discussion. It is the uniform mode, in such compends as I have seen, to show first that every line joining two inflexions meets the cubic again in a third inflexion; second, that through the nine inflexions there must lie in all twelve such lines; and thirdly, that three lines can be selected which contain all nine inflexions. These three lines are termed an inflexional triangle, and the entire twelve are thought of as constituting four inflexional triangles. But there is another arrangement of the nine lines remaining after the erasure of one inflexional triangle, which I have not happened to find mentioned, which yet seems the easiest and most natural mode of access to the inflexional triangle itself.

It shall be presupposed known that there are nine inflexional points, and that every line joining two of them contains also a third. Select two inflexional points $A$, $B$, and any third $C$ not collinear with the first two. Call these three an *inflexional triplet*. Join them by three lines, and produce $BC$, $CA$, $AB$ to meet the cubic in a second inflexional triplet, in the points $A_1$, $B_1$, $C_1$ respectively.*

Repeating the process upon these three, determine a third triplet $A_2$, $B_2$, $C_2$. From these, determine similarly a fourth triplet. Since its points cannot be additional inflexions, nine having been included already; and since they cannot be the points of the second triplet (as is evident from the figure) unless certain inflexions coincide, they must be the

*It is easily seen that $A_1$, $B_1$, $C_1$, and again $A_2$, $B_2$, $C_2$, are not collinear.

points of the first triplet. Repetition of the process would lead us through the same period of three triplets. We have seen accordingly that: *The nine inflexional points of a plane cubic may be regarded as vertices of three triangles, each of which is inscribed in one and circumscribed about another, the three thus forming a closed series.*

Recalling the exact order followed in the construction of the diagram, we notice that it contains no line joining $A_1A$ or $A_1A_2$, but that the three lines meeting in $A_1$ do contain all inflexional points except $A, A_2$. Hence $\overline{A_1A}$ must contain $A_2$, and similarly $\overline{B_1B}$ contains $B_2$, $\overline{C_1C}$ contains $C_2$. (Of course a *real* diagram cannot show this.) Hence *the three inflexional lines not appearing in the diagram contain all nine points, and constitute by definition an inflexional triangle.* The three other inflexional triangles appear easily, from the notation employed. They contain respectively the lines:

$$\left\{\begin{matrix}\overline{AB}\\ \overline{A_1B_1}\\ \overline{A_2B_2}\end{matrix}\right\}, \quad \left\{\begin{matrix}\overline{AC}\\ \overline{A_1C_1}\\ \overline{A_2C_2}\end{matrix}\right\}, \text{ and } \left\{\begin{matrix}\overline{BC}\\ \overline{B_1C_1}\\ \overline{B_2C_2}\end{matrix}\right\}.$$

Starting with any three inflexions, $A$. $B$, $C$, which do not lie in a straight line, we have found directly a closed chain of three triplets, and indirectly one (excluded) inflexional triangle. Each triplet of the three would evidently lead to the same closed chain, and to the exclusion of the same inflexional triangle. But of such triplets there exist in all $\frac{9\cdot8\cdot6}{1\cdot2\cdot3} = 72$; and of closed chains, $\frac{72}{3} = 24$, while there are but 4 inflexional triangles. Therefore each of the four is excluded by 6 different closed chains of triplets. And our diagram shows this; for the same diagram results if we start from any of the five other triplets lying on triangles having $A$ for a vertex: $AB_1C_1$, $ABC_2$, $AC_1B_2$, $ACB_2$, $AB_1C_2$. Since the diagram exhibits the nine lines belonging to three inflexional triangles, the same facts might have been stated thus: *The lines which constitute three inflexional triangles of a plane cubic may be regarded in six different ways as constituting a closed series of three triangles, each inscribed in the preceding (and circumscribed about the following).*

We may add: *In these arrangements each side of one inflexional triangle occurs four times in a triplet with each side of every other inflexional triangle.* This would be plain also

from the four inflexional lines that meet, *e. g.*, $\overline{AB}$ and $\overline{AC}$, in inflexional points other than $A$.

The nonsingular plane cubic is one of an indefinitely great series of "elliptic normal curves" in spaces of three dimensions, four dimensions, etc., respectively. On each of these there exists a configuration analogous to that of the inflexion-system of the cubic. All such are mere matters of course when the points of the curve are represented by values of an elliptic integral of the first sort. But they are no less easy of discovery by the immediate extension of the foregoing method; and the closed chains of four tetrahedra, five pentahedra, etc. appear to be novel and interesting objects for the geometric imagination. Further, by the application of elliptic parameters to these objects an extensive theory may be evolved, peculiar and not devoid of profit.

NORTHWESTERN UNIVERSITY,
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## ON THE INTERSECTIONS OF PLANE CURVES.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

LINEA Ordinis ($n$) occurrere potest aliæ ejusdem Ordinis in punctis $n^2$. Proinde duæ Lineæ Ordinis ($n$) per eadem puncta $n^2$ transire nonnunquam possunt; adeoque puncta data quorum numerus est $\frac{1}{2}(n^2 + 3n)$ non sufficiunt ad Lineam Ordinis ($n$) ita determinandam ut unica sit curva quæ per ea data puncta duci possit: Cum vero coefficientes in æquatione generali ad Lineam Ordinis ($n$) sint $\frac{1}{2}(n^2 + 3n)$, patet si plura dentur puncta, Lineam Ordinis ($n$) per ea forsan duci non posse et Problema reddi posse impossibile. Sic novem puncta non adeo plene determinant Lineam Ordinis tertii ac quinque Lineam Ordinis secundi, decem tamen ad Lineam tertii Ordinis determinandam nimia sunt.

MACLAURIN, *Geometria Organica*, 1720; Sect. V, Lemma III, Corol: II; p. 137.

Ensuite je ferai voir le défaut, qui se trouve dans ces conséquences, qui consiste dans une fort subtile précipitation du raisonnement, laquelle n'etant pas si facile à découvrir, nous doit rendre extrémement circonspects, principalement dans les autres sciences, afin que nous ne nous laissions pas séduire par de semblables contradictions apparentes. Car,

si dans la Géométrie nous sommes assujettis à des difficultés si remarquables, où il est pourtant permis de ramener toutes les idées presque au plus haut degré de justesse, combien plus pourrons-nous être embarrassés dans les autres sciences, où il n'est pas possible de parvenir à des idées assez précises, et où il est infiniment plus difficile de se garantir de pareilles fautes dans le raisonnement?

EULER, *Sur une contradiction apparente dans la doctrine des lignes courbes*, 1748–50, p. 221.

Cette contradiction se léve par la Remarque qui termine le § 38. C'est qu'encore qu'on ait autant d'équations qu'il en faut, généralement parlant, pour déterminer tous les coëfficients de l'équation prise pour réprésenter la Courbe qui doit passer par un certain nombre de points donnés, il peut pourtant arriver que ces coëfficients restent indéterminés. Alors l'équation prise reste indéterminée et réprésente une infinité de Courbes du même Ordre. * * * Ce qui est un véritable paradoxe.

CRAMER. *Introduction à l'Analyse des Lignes Courbes algébriques*, 1750; Ch. III, § 48, p. 79.

In mathematischen Dingen kann nicht lange etwas ein Paradox bleiben, eben so wenig als ein Kunstgriff lange als solcher sich behaupten kann; das Paradox verschwindet, wenn verdeckte Mittelglieder der Verkettung mathematischer Sätze hervortreten und der Kunstgriff verliert sich in einer neuen durchgreifenden Behandlungsweise.

PLÜCKER. *Theorie der Algebraischen Curven*, 1839; p. 12.

Es dauerte einige Zeit, bis dieses so genannte Paradoxon seiner wahren Bedeutung nach, nämlich als Quelle von Sätzen. erkannt wurde.

CLEBSCH. *Zum Gedächtniss an Julius Plücker*, 1871.

The theory of the intersections of curves has probably led its investigators into more errors than any other modern theory. Even the history of the central question, the so-called Cramer paradox. is usually given incorrectly, with the omission of all reference to Maclaurin. This is the more surprising, inasmuch as Cramer himself ascribes to Maclaurin the theorem that the $n^2$ intersections of two curves of order $n$ impose only a certain number of conditions, and gives the exact reference. Plücker was familiar with the passage in Cramer; nevertheless he overlooks Maclaurin, though he was sufficiently interested historically in the Cramer paradox to give also the reference to Euler.

He brings his account up to date with a mention of his own "Théorèmes généraux concernant les équations d'un degré quelconque entre un nombre quelconque d'inconnues," *Crelle's Journal*, vol. 16, pp. 47–57, 1837, and Jacobi's "De relationibus, quæ locum habere debent inter puncta intersectionis duarum curvarum vel trium superficierum algebraicarum dati ordinis, simul cum enodatione paradoxi algebraici," *Crelle's Journal*, vol. 15, pp. 285–308, 1836, stating that both memoirs were in the editor's hands at the same time. Clebsch, in his "Gedächtnissrede," though speaking from minute first-hand knowledge of Cramer, makes no mention of Maclaurin; in Clebsch-Lindemann's Vorlesungen über Geometrie, 1876, p. 426, and in Salmon's Higher Plane Curves, 3d ed., 1879, p. 22, the account is substantially that of Plücker, and thus Maclaurin is ignored here, as also in Loria's Il passato ed il presente delle principali teorie geometriche, 2d ed., 1896. Brill and Noether, however, supply the reference in their Bericht, "Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit," 1894, *Jahresberichte der Deutschen Mathematiker-Vereinigung*, vol. 3, p. 130; and as it is given also in the analysis of the Geometria Organica in Cantor's Vorlesungen über Geschichte der Mathematik, vol 3 (Part 2) 1896, pp. 426, 427, Maclaurin will probably obtain due credit on this score in future.

Maclaurin's precise understanding is that the $\frac{1}{2}n(n+3)$ points assigned for the determination of a curve of order $n$, if included in the $n^2$ points given by the intersections of two curves of order $n$, will fail to determine the curve uniquely; he makes no attempt to explain it. Cramer goes further than this; he gives the explanation, and says that an infinity of curves can be found satisfying the given conditions, a result arrived at by algebraic considerations. He does not formulate the conclusion that a certain number of the nine or sixteen points (he speaks principally of cubic and quartic curves) are a necessary consequence of the remainder, though Clebsch, erroneously according to Brill and Noether, assigns this view of the theorem to him. The actual wording does not disprove his knowledge of this, nor does it necessarily imply it; but it is plain that the fact, even if he noticed it, did not strike him as the important one in the theorem. Naturally he does not recognize the linear nature of the system of curves that can be found satisfying the given conditions, nor does he realize that only one indeterminate quantity is involved, as may be clearly seen on p. 60; this particular step seems to

have been taken by Lamé, who expressed the system of curves in the form $m E + m' E' = 0$, Examen des différentes méthodes employées pour résoudre les problèmes de géométrie, 1818, p. 28. (See also the Brill-Noether Bericht, pp. 289 et. sq., and Plücker's Collected Papers, vol. 1, p. 597, Schoenflies, for other historical references.)

Euler's resolution of the paradox, though published in the same year as that of Cramer, was communicated to the Berlin Academy two years earlier. It agrees with that of Cramer, and goes just as far, or even a little further, if we give the most liberal interpretation to a sentence on p. 232. Euler is decidedly lavish of explanation, with his Propositions I, II, Difficulties I, II, III, and Reflexions I, II, III, IV, all preliminary to the illustrations by means of which he explains his point. The memoir bears the title "Sur une contradiction apparente dans la doctrine des lignes courbes," *Mémoires de l'Academie Royale des Sciences et Belles Lettres*, Berlin (1748), vol. 4, 1750, pp. 219–233.*

The allied theorem that if of the $n^2$ intersections of two curves of order $n$, $np$ lie on a curve of order $p$, the remainder lie on a curve of order $n - p$, was given by Gergonne, *Annales de Mathématiques*, vol. 17, p. 220, 1827. In vol. 19 of the same periodical there is a short memoir by Plücker, "Recherches sur les courbes algébriques de tous les degrés," in which the algebraic view of the Cramer paradox, namely, that from a given system of equations others of the same type are derivable, is applied to the theory of conjugate points with respect to a conic. A more important contribution to the theory is contained in the Analytisch-geometrische Entwickelungen, vol. 1, 1828, p. 228 (foot note), where one curve through the points is regarded as fixed. Clebsch speaks of this as an indispensable preliminary to any advance in the theory; Brill and Noether, in reproducing this judgment, p. 290, modify it somewhat, simply emphasizing the importance of the convention for the theory of functions, with a tacit recognition of the fact that the significance of the paradox is not confined to this.

The investigations associated with this "Quelle von

* This memoir is immediately followed by the "Démonstration sur le nombre des points où deux lignes des ordres quelconques peuvent se couper," pp. 234-248, in which the number of intersections, $mn$, arrived at by induction by Maclaurin, is proved by the process of elimination by symmetric functions. This is the only one of the two memoirs explicitly mentioned in the Brill-Noether Bericht, pp. 140, 143; the first is, however, the one referred to, by content not by title, on pp. 138, 289, though the cross-reference to p. 138 on p. 143 would lead one to suppose that the second memoir is intended.

Sätzen" fall into three divisions, which for convenience will be referred to as German, Italian, and English. The first is virtually the Brill-Noether work, originating with the memoir "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie," *Mathematische Annalen*, vol. 7, pp. 269–310, 1873, in which the consequences, immediate and remote, are considered in the light of the theory of functions. The one curve, fixed by Plücker, is primarily the locus in quô; on this certain groups of points are selected by purely analytical characteristics which they have in common; the subject of investigation is the system of groups of points so determined, and it is by a rather grudging concession that these are spoken of as cut out by adjoint curves; notwithstanding the important geometrical applications of the theory, geometry is little more than an accessory after the fact.

In the Italian work, on the contrary, as exhibited in the series of memoirs by Caporali, Jung, Bertini,* Guccia, Segre, Castelnuovo, and others, the geometrical view is paramount. The subject of investigation is the linear system of curves of any assigned order, determined by given fixed points; in intimate connection with this there are certain derived systems. While the groups of points cut out by the system on any one curve—fixed only for the moment in this class of investigations—do in fact present themselves, yet they are of quite secondary interest.

It is not easy to give a concise description of the investigations of the third division. They concern themselves hardly at all with a system of groups of points on one curve, hardly at all with a system of curves through one group. The locus in quô is the plane; the subject of investigation is the group of points, the complete or partial intersection of two curves of the same or different orders, regarded as a means of imposing conditions on another curve of any order. In speaking of this as the English theory, I do not wish to imply any geographical restriction, but simply to emphasize the fact that of the three classes of investigation, this is the one chiefly recognized in English works. (See Salmon's Higher Plane Curves, §§ 28–34.) Hitherto it has been hardly more than a collection of detached theorems, of the type of Cayley's well-known intersection theorem, growing directly out of the simpler theorems of Maclaurin on the

* The greater part of one of Bertini's memoirs, "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico," *Annali di Matematica*, Ser. II., vol. 22, pp. 1–40, 1894, is devoted to an exposition of the Brill-Noether theory, and hence belongs to the first division.

intersections of curves, these theorems, in virtue of the simplicity of their derivation, proving immediately fruitful in applications to geometry.* Inasmuch as the groups of points that occur are connected by a curve through two or more, there are relations to the German theory; inasmuch as the two or more curves that are passed through a group may be taken of the same order, so belonging to a linear system of curves, there is something in common with the Italian theory; and coincidences of results are to be expected. That the various detached theorems are of importance, not only in their direct geometrical application, but also as products of a process of value, has, I think, been abundantly proved by the latest memoir in this line, "Point-Groups in relation to Curves," by Mr. F. S. Macaulay, in the *Proceedings of the London Mathematical Society*, 1895, vol. 26, pp. 495–544.

Cayley's theorem, that a curve of order $n$ through all but $\frac{1}{2}(l+m-n-1)(l+m-n-2)$ of the points common to two curves of orders $l$ and $m$ will necessarily pass through the remainder if $n$ be not less than $l$ or $m$ but less than their sum (Collected Papers, vol. 1, no. 5), was given in a short paper in the *Cambridge Mathematical Journal*, vol. 3, 1843. Clebsch in 1871 ventured the opinion that with this the algebraic side of the investigation of the Cramer paradox can be regarded as closed. But this was disproved ten years later, when Bacharach, in his inaugural dissertation, "Ueber Schnittpunktsysteme algebraischer Curven" (Erlangen), incorporated in the memoir "Ueber den Cayley'schen Schnittpunktsatz," *Mathematische Annalen*, vol. 26, pp. 275–299, 1886, showed that a specialization in the position of the $\delta$ points $[\delta = \frac{1}{2}(l+m-n-1)(l+m-n-2)]$, would affect the theorem. To the recognition of this possible specialization belongs the credit of much of the later geometrical work. Bacharach's form of the theorem is:—a curve of order $n$ through $lm-\delta$ of the points will of necessity pass through the remaining $\delta$, unless these lie on a curve of order $l+m-n-3$. This modification was discussed and adopted by Cayley in the memoir "On the Intersections of Curves" (Collected Papers, vol. 12, no. 868), *Mathematische Annalen*, vol. 30, 1887; a more detailed proof

*See for example Cremona, Introduzione ad una teoria geometrica delle curve piane, 1862, p. 39; Olivier, in *Crelle*, vols. 70, 71, 1869, 1870; Study, *Math. Ann.*, vol. 36, 1890. In this connection it is perhaps not superfluous to call attention to Bacharach's remark, *Math. Ann.*, vol. 26, p. 295, that the professed extensions of Olivier's theorems given on pp. 762–764 of Clebsch-Lindemann are not legitimate; it may be added that the whole of Sect. V. of Chap. 6, pp. 753–764, needs correction.

was given by Zeuthen, "Sur la détermination d'une courbe algébrique par des points donnés," *Mathematische Annalen*, vol. 31, 1888. It is necessary, in considering the series of theorems of which this is the type, to keep the two parts distinct (see Brill-Noether Bericht, p. 290). The expression for the postulation, to use Cayley's term, of the $lm$ points, namely,

$$D = lm - \delta = lm - \tfrac{1}{2}(l + m - n - 1)(l + m - n - 2),$$

is true without exception; but it is not true without exception that the $\delta$ points necessarily follow from the $D$ points. It may be the case that the $lm$ points consist of $D_1$, $< D$, with an induced number $\delta_1$, and other points. Thus for example the sixteen intersections of two quartics, one of which consists of a cubic and a straight line, are not such that any three follow from the thirteen; from eleven of the intersections of the cubic and quartic the twelfth follows; from these twelve with two (arbitrary) points, the fifteenth and sixteenth follow.

Mr. Macaulay introduces the convenient term *n-ic excess* for the excess of the number of points in any one group over the number of independent conditions that these points impose on $C_n$, a curve of order $n$. Thus if $N$ be the number of points, $D$ their postulation for $n$-ics, the $n$-ic excess, $r_n$, $= N - D$. In Castelnuovo's "Ricerche generali sopra i sistemi lineari di curve piane," *Mem. della R. Acc. d. Sc. di Torino*, 1891, Ser. II., vol. 42, the analogous term *sovrabbondanza* is used with reference to the system of curves of any order determined by the points; the system is said to be *sovrabbondante;* if there is no excess, the system is *regolare.* By the *n-ic defect* of the points, $q_n$, is to be understood the number of degrees of freedom of a curve of order $n$ through the points; hence

$$N - r_n + q_n = \tfrac{1}{2}n(n + 3).$$

The statement of the first part of the Cayley theorem is then that if $n$ be not less than $l$ or $m$ the $n$-ic excess of $I(l, m)$* is

$$[\tfrac{1}{2}(l + m - n - 1)(l + m - n - 2)],$$

where the square brackets indicate that the product so given presents itself only when the individual factors are positive. Thus if $n \geq l + m - 2$, the $n$-ic excess is zero. The term

* I use this symbol to denote the complete intersection of two curves of orders $l$, $m$.

*multiplicity* represents the *Mannigfaltigkeit* of the German work, that is, the index $q$ in the usual symbol $g^q_Q$; it is shown that the multiplicity of the system of groups cut out on a base-curve of order $m$ by curves of order $n$ through a group $N$ on the base-curve is

$$q_n - [\tfrac{1}{2}(n-m+1)(n-m+2)].$$

In the course of the work the interesting fact is emphasized that if a group of points has an excess for a curve of any one order it has in general an excess for curves of other orders; and that the $n$-ic excess for a given group diminishes as $n$ increases. The German and Italian investigations have not required the systematic consideration of curves of all orders through one group of points, so apparently the use that can be made of this fact has not been realized. It is shown that given the number of points, and the successive values of $r_n$, which together form the characterization of the group, its construction can be found. Such a group, one which owing to some interconnection of the points has an excess for at least one curve, is said to be *of special form*, this particular phrase being used because the simpler term special * is already appropriated in the German and Italian investigations.

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* In the footnote to p. 499 Mr. Macaulay gives only the older use of the word special; this was originally applied to groups $G^q_Q$ or systems $g^q_Q$ in which $Q-q<p-1$; see Brill and Noether, *Math. Ann.*, vol. 7, p. 280; Clebsch-Lindemann, pp. 686, 699; Brill-Noether Bericht, p. 338. The later and more convenient usage, recognizing the more important ground of distinction in invariantic properties, makes special apply when

$$Q-q \leq p-1,$$

with some additional epithet for the restricted case $Q-q<p-1$; see Fiedler's translation of Salmon's Higher Plane Curves, 2d ed., 1882, p. 418; Castelnuovo, "Ricerche di geometria sulle curve algebriche," *Atti della R. Acc. d. Sc. di Torino*, 1889, vol. 24, § 23; also "Ricerche generali," Cap. II., § 18; Bertini, "La geometria delle serie lineari," § 5, 21; Brill-Noether Bericht, pp. 359, 542.

To a group of points which imposes fewer conditions on a curve of assigned order than would be supposed from the number of the points, Cayley's "special system," Collected Papers, vol. 7, p. 254, the term *ausgezeichnete* is applied (Brill-Noether Bericht, p. 543), the epithet having reference to a particular derivation of such a group; thus Mr. Macaulay's point-group of special form is ausgezeichnete. Such special groups as are also ausgezeichnete, that is, groups for which $Q-q<p-1$, are spoken of as *eigentlich* special (Fiedler, p. 418, Brill-Noether Bericht, p. 547). I am not aware that any English equivalent has been suggested for eigentlich special, but one will surely be needed. Seeing that the term is applied to a class of cases, $Q-q<p-1$, included in the more extensive class special, $Q-q\leq p-1$, *intraspecial* appears suitable.

If a group contains a point-group of special form, with other points that have no connection with this, it is said to be *redundant;* if while containing only a part of a point-group of special form, it contains so much as to oblige a curve to pass through the rest, it is *incomplete;* if it is neither redundant nor incomplete, it is *complete.* On this the criticism may be made that the completeness, redundancy, or incompleteness really depends on the order of the curve, instead of being, as the wording would imply, an intrinsic property of the point-group. Thus for example the 30 intersections of two curves of orders 5 and 6 impose 24, 27, 29 conditions on curves of orders 6, 7, 8; taking 28 of these 30 points, we have a group which is incomplete with reference to curves of orders 6 and 7, but not with reference to curves of order 8. Moreover, a group may be at one and the same time both redundant and incomplete for any one curve, as for example the group of 14 points composed of the 12 intersections of a cubic and a quartic and 2 arbitrary points. The 12 intersections are a point-group of special form, which by the addition of the 2 arbitrary points is made redundant; but any quartic through the 14 will pass through other 2 fixed points, not derivable from the 2 alone; the group is therefore incomplete. This criticism, however, does not affect the general argument.

The process of proof is in general that of Zeuthen (l. c.). This is applied in the first few sections to establish the known theorems:

I. the $l$-ic excess of $I(l, m) = \frac{1}{2}(m-1)(m-2)$, where $l \geq m-2$;

II. any curve of order $n$ through $I(l, m)$ is of the form $C_l S_{n-l} + C_m S_{n-m} = 0$, (Noether's fundamental theorem);

III. the theorem of Residuation;

IV. the necessary and sufficient condition that a point-group on a $C_m$ lie on a $C_{m-3}$ is that the multiplicity be $\geq N - \frac{1}{2}m(m-3)$ with the corresponding result, equivalent to $Q - q \leq p - 1$, for curves with multiple points.

In § 18 the Riemann-Roch theorem is given in the form:—the multiplicity of any point-group on $C_m$ is equal to its $(m-3)$-ic excess. The Brill-Noether form of the theorem is expressed by a reciprocal relation between the defects and excesses, namely, if a point-group $N$ lie on a curve of order $m-3$, and have then a residual $N'$, the $(m-3)$-ic defect and excess of $N$ are respectively equal to the $(m-3)$-ic excess and defect of $N'$; and the difference of the numbers $N$, $N'$ is equal to twice the difference of the multiplicities.

Among other results in § 19 there are two of particular importance for the following sections; they are obtained with the help of the Riemann-Roch theorem. (§ 19, iii.) If $N + N' = I(m, n)$, and the $n$-ic excess of $N$ be not zero, then $N'$ lies on a $C_{m-3}$ which does not pass through $N$; and if the $n$-ic excess of $N$ be zero, $N'$ does not lie on a $C_{m-3}$; unless $n \leq m-3$, in which case any $C_{m-3}$ through $N'$ necessarily passes through $N$. This is connected with the Bacharach version of the Cayley theorem. Writing $m = n$ it becomes the known theorem, stated in its most convenient form by Castelnuovo (Ricerche generali):—a system is regular or *sorrabbondante* according as the characteristic series is non-special or special. § 19, iv, the generalized form of no. iii, appears to be a new result, or, at any rate, if not absolutely new, it is shown in a new and more significant light. If $N + N' = I(l, m)$, the lowest order of a curve that passes through $N'$ without passing through $N$ is known when we know the highest order, $n$, of a curve through $N$ for which $N$ has an excess; namely, it is $l + m - n - 3$. Obviously curves of all higher orders can be passed through $N'$ without passing through $N$, and it has been seen that for all lower values of $n$ the group $N$ will have an excess; hence what the theorem really shows is that according as $r_n > 0$ or $= 0$, a curve of order $l + m - n - 3$ can or cannot be passed through $N'$ without passing through $N$, and conversely.

In theorem V, § 20, the $(l + m - n - 3)$-ic excess and defect of $N'$ are expressed in terms of the $n$-ic defect and excess of $N$ in two equations, either of which is deducible from the other; these are

$$\begin{aligned} r'_{l+m-n-3} &= q_n + 1 - [\tfrac{1}{2}(n-l+1)(n-l+2)] \\ &\quad - [\tfrac{1}{2}(n-m+1)(n-m+2)], \\ q'_{l+m-n-3} &= r_n - 1 + [\tfrac{1}{2}(l-n-1)(l-n-2)] \\ &\quad + [\tfrac{1}{2}(m-n-1)(m-n-2)]. \end{aligned}$$

Writing $n'$ for $l + m - n - 3$, so that $n + n' = l + m - 3$, these relations become

$$\begin{aligned} r_n' &= q_n + 1 - [\tfrac{1}{2}(n-l+1)(n-l+2)] \\ &\quad - [\tfrac{1}{2}(n-m+1)(n-m+2)], \\ r_n &= q_n' + 1 - [\tfrac{1}{2}(n'-l+1)(n'-l+2)] \\ &\quad - [\tfrac{1}{2}(n'-m+1)(n'-m+2)]. \end{aligned}$$

When $l = n = n' = m - 3$, these reduce to the reciprocal relations already proved, namely $r'_{m-3} = q_{m-3}$, $q'_{m-3} = r_{m-3}$;

the theorem is, in fact, an extension of the Riemann-Roch theorem on the geometrical side.

If $n$ be less than both $l$ and $m$, and consequently $n'$ be greater than both $l-3$ and $m-3$, the first of the formulæ becomes

$$r_{n'}' = q_n + 1;$$

and if $n$ be greater than both $l-3$ and $m-3$, so that $n'$ is less than both $l$ and $m$, the second becomes

$$r_n = q_{n'}' + 1,$$

that is,

$$q_{n'}' = r_n - 1.$$

If the difference of $l$ and $m$ is less than 3, one or other of these two cases necessarily presents itself. If the difference of $l$ and $m$ is greater than or equal to 3, a sufficiently great value of $n$ ($> l-3$ where $l > m$), or a sufficiently small value ($< m$), with the accompanying values of $n'$ ($< m$ or $> l-3$), will ensure simplification in one or other of the formulæ.

In the course of the proof of this theorem, Mr. Macaulay establishes the second part of the Cayley theorem, in the form of a criterion for deciding whether the $C_n$ that is drawn through $N$ of the intersections of $C_l$, $C_m$, passes through the remaining $N'(=lm-N)$ points; the $C_n$ does, or does not, pass through these $N'$ points according as the $(l+m-n-3)$-ic excess is, or is not, zero. In the use of this criterion a remark made in § 7 is of importance in determining the excess; the greatest number of points that can supply independent conditions for curves of order $k$ is $\frac{1}{2}(k+1)(k+2)$, seeing that this number would oblige all the coefficients in $C_k$ to vanish. Hence when $N > \frac{1}{2}(k+1)(k+2)$, the $k$-ic excess of $N$ general points is $N - \frac{1}{2}(k+1)(k+2)$.

Having found in any given case that the $N$ points which form the partial intersection of $C_l$ and $C_m$ have an $n$-ic excess $r_n$, is there any way of deciding whether a $C_n$ through $N - r_n$ of these necessarily passes through the remainder? Mr. Macaulay does not raise this question, but it is irresistibly suggested by the Cayley-Bacharach theorem. The fear of exhibiting a "subtile précipitation de raisonnement" almost dissuades one from attempting to give an answer offhand, though it appears highly probable that this depends solely upon the curve of order $l+m-n-3$ through the $lm-N$ points, which may, or may not, necessarily pass through some of the $N$ points; those through which it necessarily passes are probably redundant in $N$ as regards the $C_n$.

By means of theorem V we can pass from the character-

ization of a group $N$ to that of a residual group. On this Mr. Macaulay bases his interesting process, §§ 26–29, for constructing a point-group of given characterization, for example, a group of 67 points whose excess for curves of order 9, 10, 11, 12 shall be 13, 7, 3, 1. The corresponding defects are 0, 5, 13, 24; hence only one $C_9$ can be passed through the points, they form part of the intersection of $C_9$ and $C_{10}$, and the residual group contains 23 points. As 12 is the highest order for which the 67 points have an excess, the $n$ of the formula $l+m-n-3$ is 12, and the lowest order of a curve through the residual 23 is consequently 4. Hence by theorem V,

$$q_4' = r_{12} - 1 = 0, \quad q_5' = r_{11} - 1 = 2,$$
$$q_6' = r_{10} - 1 = 6, \quad q_7' = r_9 - 1 = 12,$$

from which

$$r_4' = 9, \quad r_5' = 5, \quad r_6' = 2, \quad r_7' = 0.$$

That $q_4'$ is zero agrees with the obvious fact that there is only one $C_4$ through the 23 points; and the value 2 for $q_5'$ shows, what again is otherwise evident, that the only $C_5$'s are those formed by the $C_4$ and any straight line. The value 6 for $q_6'$ shows that there are proper sextics through the points; hence the residual group $N'$ is 23 of the 24 intersections of $C_4$ and $C_6$; and a group $N$ of the desired characterization is constructed by passing $C_9$ and $C_{10}$ through 23 of the intersections of $C_4$ and $C_6$.

The general process requires that the lowest orders of curves through $N$ be known, $l$ and $m$, where $l \geq m$; if the complete characterization is given, this condition is fulfilled. If $q_m > 0$, $l = m$. Let $n$ be the highest order for which the given points have an excess, $n \geq l \geq m$; the $C_l$ and $C_m$ through $N$ determine a residual $N'$; the curve of lowest order through $N'$ is $C_{l+m-n-3}$, and curves of all higher orders will pass through $N'$. The defects are known from theorem V in the form

$$q'_{l+m-n-3} = r_n - 1, \quad q'_{l+m-n-2} = r_{n-1} - 1, \cdots;$$

from these $r'_{l+m-n-3}$, etc., are known; and it is shown that $N'$ cannot have an excess for a curve of order greater than $m-3$.

Thus from the characterization of $N$ the characterization of $N'$ is known; and $N'$ is less complex than $N$. This is merely stated in the text, but a formal proof, though simple, is hardly superfluous. It is to be shown that $N' < N$, $l' < l$, $m' < m$, $n' < n$. As $N + N' = lm$, it must be shown

that $N > \frac{1}{2} lm$. Now if $N \leq \frac{1}{2} lm$, a $C_{l-1}$ could be passed through $N$; for the postulandum of $C_{l-1}$ is $\frac{1}{2}(l-1)(l+2)$, and this is $\geq \frac{1}{2} lm$ if $l^2 + l - 2 \geq lm$, that is, if $l(l-m+1) \geq 2$. As $l \geq m$, and $l \geq 2$, this is satisfied. Thus a $C_{l-1}$ could be passed through $N$, and if $l > m$, more than one such curve, contrary to the hypothesis that $m$ and $l$ are the lowest orders. Hence $N > \frac{1}{2} lm$, $N' < \frac{1}{2} lm$ and therefore $N' < N$. Again, it is shown that $n' \leq m - 3$, hence $n' < m$, and therefore $n' < n$. Also $l'$ and $m'$ are both $\leq n'$, that is, $< m$, and consequently $< l$. Thus the residual group $N'$ contains fewer points than the given one, it is the partial intersection of curves of lower order than any that pass through the given one, and the conditions imposed by it become independent at an earlier stage; it is therefore less complex than the given one. Similarly from $N'$ we pass to a simpler group $N''$, and so on.

Any obvious impossibility indicates the non-existence of a point-group with the given characterization; *e. g.*, an attempt to find* $N = 369$ with excess 48, 28, 16, 7, 2 for curves of order 24, 25, 26, 27, 28, leads to $N''' = 18$, with defects 3, 7, 17 for curves of order 5, 6, 7. Now 18 points, even if arbitrary, would leave $C_6$ with $27 - 18 = 9$ degrees of freedom; hence necessarily $q_6''' \geq 9$, and the calculated value 7 indicates impossibility. Attention would be drawn to this by the resulting negative value, $-2$, for $r_6'''$

Of course what is really proved is that the existence of a point-group of the specified type depends on the existence of $N'$, this on the existence of $N''$, and so on; thus in every case the last group obtained must be examined with care. This last group, however, will necessarily be of a fairly simple construction; for so long as there are excesses, we can reduce by the given process; and after each step in the reduction the order of the highest curve involved is less by three or more than the order of the lowest curve in the preceding stage.

The process must be applied with due regard to certain conditions, pointed out in §§ 27, 29; but even with these limitations it is a valuable contribution to the theory of point-groups in a plane. It might with advantage have been brought more prominently forward, with a rather more emphatic statement; as it stands, its importance may be overlooked by a rapid reader. For instance, the reviewer in the *Jahrbuch über die Fortschritte der Mathematik*, Jahrgang 1895, vol. 26, p. 653, says that the author does not succeed in finding a general law for $r_n$; that all that he can show is

* This is a modification of one of Mr. Macaulay's illustrations.

that the value of $r_n$ cannot exceed certain limits. As a matter of fact, the principal thing in Mr. Macaulay's paper is that he *does* give the law for $r_n$. For if the points are entirely arbitrary, $r_n$ is in general zero; if they are not arbitrary, something must be given about them; if this be the construction, then these theorems give the values of $r_n$, and if the values of $r_n$ are given, this process shows the construction.

I have spoken only of the sections that relate to non-singular curves. The memoir professedly deals also with curves endowed with multiple points of any order, provided all the tangents are distinct; for these multiple points a convention is adopted which strikes me as unsatisfactory. It would doubtless be convenient if we could describe the nature of a multiple point, with tangents distinct or coincident, in such a way that for all purposes its effect should be the simple sum of the effects of its components. As regards the effect on the class and deficiency, this has been accomplished in ways both geometrical and analytical; but it has not been done, for example, for the effect on the Jacobian of a net of curves, nor even for the effect on the Hessian of a single curve. Mr. Macaulay's convention of "clusters," §§ 1, 8, etc., does not seem to me to take us one step further in this line; and his replacement of a curve at a multiple point by a penultimate form, regarding this as an infinitely small curve of definite, though to a certain extent arbitrary, shape—in order to deal with the intersection of two curves at a common multiple point where all the tangents are distinct—is, in my opinion, no advance on the accepted approximation to the curve by means of its tangents.

There are other matters of interest in the memoir, which occupies fifty pages; but they do not come within the scope of this note, in which I have simply tried to give a general account, in what seems to me its proper perspective, of the most striking part of an interesting and valuable paper. Many of the results contained are doubtless well known; but so far as I am able to judge, there is more that is new than the reviewer in the *Jahrbuch* is disposed to acknowledge. In this particular line of work the various theorems proceed so directly from their common source that when once recognized a slight twist given to a formula is all that is needed to prove many of them. The real importance of Mr. Macaulay's work, however, consists in the fact that it is a systematic treatment by purely geometrical methods of the interdependence of curves and determining points.

BRYN MAWR COLLEGE,
*January*, 1898.

## EULER'S USE OF $i$ TO REPRESENT AN IMAGINARY.

In the *Introductio in Analysin Infinitorum*, Lausannæ, 1748, Euler uses $i$ to represent an infinitesimal, an infinite, and a positive integer, but in Tom. II, p. 290, we find "Cum enim numerorum negativorum Logarithmi sint imaginarii, * * erit $l. -n$, quantitas imaginaria, quæ sit $= i$."

In a paper *De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet, M.S. Academiae* [Petropolitanae] *exhibit. die 5. maii* 1777, reproduced in Vol. IV., *Institutiones Calculi Integralis*, Petropoli, 1845, pp. 183–194, there occur such passages as "** formulam $\sqrt{-1}$ littera $i$ in posterum designabo, ita ut sit $ii = -1$, ideoque $\frac{1}{i} = -i$." "* * loco cos.$\varphi$ has duas partes substituamus

$$\tfrac{1}{2}(\cos.\varphi + i\sin.\varphi) + \tfrac{1}{2}(\cos.\varphi - i\sin.\varphi).$$

"Constat autem esse

$$(\cos.\varphi + i\sin.\varphi)^n = \cos. n\varphi + i\sin. n\varphi."$$

"** ubi tam $x$ quam $y$ imaginaria involvit, hanc ob rem ponamus brevitatis gratia $x = r + is$, $y = r - is$."

These extracts would seem to dispose of the claim that "Gauss introduced the use of $i$ to represent $\sqrt{-1}$." (See Baltzer, Fink, Wolf, Holzmüller, Thomae, Suter, Harnack, Durège, Chrystal, etc.)

W. W. BEMAN.

UNIVERSITY OF MICHIGAN.

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## NOTE ON THE ROOTS OF BESSEL'S FUNCTIONS.

VARIOUS proofs have recently been given of the theorem that between two successive positive (or negative) roots of $J_n(x)$ lies one and only one root of $J_{n+1}(x)$. The following proof (which is contained along with other investigations concerning the roots of Bessel's functions and the hypergeometric series in a paper sent last June to the *American Journal of Mathematics*) is simpler and more elementary than those heretofore given. It depends on the formulæ:

$$\frac{d\,[x^{-n}J_n(x)]}{dx} = -x^{-n}J_{n+1}(x), \quad \frac{d\,[x^{n+1}J_{n+1}(x)]}{dx} = x^{n+1}J_n(x).$$

Applying Rolle's theorem to the first of these formulæ we see that between two consecutive positive (or negative) roots of $J_n(x)$ lies at least one root of $J_{n+1}(x)$; and from the second we see in the same way that between two successive positive (or negative) roots of $J_{n-1}(x)$ lies at least one root of $J_n(x)$. Thus the theorem is proved.

M. B. PORTER.

UNIVERSITY OF TEXAS.

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## SHORTER NOTICES.

*Introductory Course in Differential Equations for Students in Classical and Engineering Colleges.* By D. A. MURRAY, PH.D., Instructor in Mathematics in Cornell University. New York, Longmans, Green, and Co., 1897. Small 8vo, pp. xv+234.

It is the aim of this work as announced in its preface, "to give a brief exposition of some of the devices employed in solving differential equations." As is becoming in a book of elementary character written with this practical end in view, no attempt is made to develop the general theory of differential equations. At the same time, the discussion of the more important cases of "solvable" equations is adequate, and the appended notes contain among other points of theoretical interest a demonstration of the "existence theorem,"—a novel feature in a treatise on differential equations, written in English.

On the whole, the book seems to be an excellent practical introduction to differential equations, containing a well proportioned and suitable treatment of most of the topics which the student needs in his first course in the subject, and of these only, a good variety of exercises, and enough historical and bibliographical notes to suggest further reading.

On the other hand it must be said that the style is not especially attractive and that certain of the discussions are not wholly satisfactory. It will suffice to cite the sections on the symbolic treatment of the linear equation with constant coefficients, which would be clearer were it shown at the outset that the operator $D$ as there involved obeys the fundamental laws of algebra; the chapter on singular solutions, in which it is not noticed that the $p$-discriminant is in the general case not a singular solution but a locus of cusps; and the demonstration in note $H$ that the necessary condition of integrability of $Pdx + Qdy + Rdz = 0$ is also sufficient.

It is unfortunate, by the way, that Boole's simple and direct demonstration of this last theorem (Differential Equations, p. 276) should not have been reproduced in the more recent text-books of the subject.

H. B. FINE.

*Elements of Theoretical Physics.* By DR. C. CHRISTIANSEN, Professor of Physics in the University of Copenhagen. Translated into English by W. F. MAGIE, PH.D., Professor of Physics in Princeton University. London, Macmillan and Co.; New York, The Macmillan Company; 1897. 8vo, pp. xii + 339. Price, $3.25.

This work by Professor Christiansen aims to give, within the limits of a single octavo volume, a survey of the whole field of mathematical physics. We realize, with the translator, that there is need of a book which will enable us to get a bird's-eye view, as it were, of the whole subject, showing the interrelations of its various branches, and using a consistent notation. The task set himself by the author in attempting to condense his survey into such a narrow compass is a most difficult one and, bearing that in mind, the measure of success which he has achieved is very gratifying. It is doubtful, however, whether in this country there is any considerable body of readers to whom this book will be of much use. It is not sufficiently comprehensive to be used as a reference book by one who has worked much in the subject, and it is too condensed and too barren of illustration and detail to be consulted with profit by the beginner.

Professor Christiansen makes no pretence of giving the various theorems in their most general form, but deals with a great many special examples and problems, and always with those of great practical importance. For example, there is no general treatment of sound, but the vibration of strings is discussed as an example of elasticity. In the chapter on thermodynamics we find Planck's equations, and the following section is given up to a discussion of their application to the subject of dissociation; but that is all we find on thermodynamic potential. The first chapter, "General theory of motion," is by far the best in the book, and can be recommended to any one wishing a digest of that subject.

This volume cannot but suffer from a comparison with Voigt's two volumes upon the same subject which appeared somewhat later. Although Voigt's aim is that of Christiansen, he has succeeded where the latter has failed, mainly,

by reason of the greater space he has allowed himself, thereby gaining enormously in comprehensiveness and lucidity. A noticeable feature of Voigt's book is the admirable bibliography of each branch of the subject; Christiansen's book has nothing of the nature of a reference for collateral reading.

Professor Magie has used for translation the German edition by Müller of the Danish original. His rendering into English is excellent, and does not slavishly follow the German text, but is free and natural.

We wish that the work of the publishers were equally deserving. The Macmillans have brought out the book in their usual elegant dress in so far as paper and type are concerned; but upon what score of economy can they be excused for the manner in which the equations are treated? One may pardon the use of the Solidus for mathematical expressions printed in the text, but surely no valid reason can be advanced for the use of the Solidus in all equations, and for the placing bodily of most of the equations in the text. The effect of a page upon the eye is dazzling and confused; and not only is eye-sight ruined, but valuable time lost, in trying to discover an equation to which reference has been made. It is to be hoped that the publishers have already come to the conclusion that this experiment in economizing in paper has not been a brilliant success.

The typographical errors which we have observed have not been numerous, and have been usually quite unimportant.

A. STANLEY MACKENZIE.

*Introduction to Infinite Series.* By WILLIAM F. OSGOOD, PH.D., Assistant Professor of Mathematics, Harvard University. To be obtained of the Publication Agent of Harvard University, 2 University Hall, Cambridge, Mass. 8vo, 71 pp. Price, 75 cents.

A beginner is almost invariably repelled by the aridity and apparent unprofitableness of the subtle discussions that occur in the theory of infinite series. In his interests it is highly desirable that stress should be laid at first on the essential principles involved, to the neglect of such parts of the subject as belong merely to what we may call mathematical technique: *e. g.*, complicated tests of convergence. That it is possible at once to interest the reader, to make no sacrifice of thoroughness, and to arrange the material in organic connection with the other parts of mathematics is proved by Professor Osgood's short pamphlet on infinite series.

The author states in the preface that his aim is to make his path easier for the beginner by "introducing him to the theory of these series in such a way that he sees at each step precisely what the question at issue is, and never enters on the proof of a theorem till he feels that the theorem actually requires proof." In carrying out this plan he makes use of diagrams and graphical illustrations, shows where possible the *a priori* probability of the theorem in question, and draws many of his illustrations from applied mathematics.

The subject matter is arranged in 5 sections: (1) convergence, (2) series as a means of computation, (3) Taylor's theorem, (4) algebraic transformations of series, (5) continuity, integration, and differentiation of series. There is also an appendix on a fundamental theorem in limits and a table of the more elementary series. Where all is good it is difficult to single out special parts for commendation. Perhaps the most valuable characteristics are the *repeated* insistence on the importance of never forgetting that the sum of an infinite series is really the limit of a sum; the distinct formation of the theorem on the existence of a limit; the distinction between Taylor's series with the remainder and the infinite series with the accompanying remark and proofs that "it is desirable that the form should be applied much more freely than has hitherto been the custom in works on the infinitesimal calculus, both because it affords a simple means of proof in a vast variety of cases and because many proofs usually given by the aid of the latter can be simplified or rendered rigorous by the aid of the former;" and finally the treatment of continuity, etc., by graphic methods in section V. The reviewer cannot agree with Professor Osgood's statement that Taylor's theorem is proved satisfactorily in all good treatises on the differential calculus.

This pamphlet is a valuable addition to the literature on convergence of series and will be of great use to those who are called upon to teach this difficult branch of pure mathematics.

J. HARKNESS.

### RECENT TEXT-BOOKS OF THE CALCULUS.

*Elements of the Differential and Integral Calculus with Applications.* By WILLIAM S. HALL, Professor of Technical Mathematics in Lafayette College. New York, D. Van Nostrand Company. 1897. 8vo, pp. xi+249.

This work is a conventional American text-book of the

better sort. In size, appearance, and contents it resembles somewhat Osborne's well-known text-book. In the first few chapters the student's attention is concentrated upon the derivative, and the fundamental formulas are deduced strictly by the method of limits. Differentials are then introduced, being defined as infinitely small increments. The method of infinitesimals is said to differ only in phraseology and notation from that of limits. Among the special features claimed for the work are the introduction of integration immediately after differentiation and before the applications, and the use of the symbol $\partial$ to denote partial differentiation. In turning over the pages we have noticed several errors or slips. The definition of limit on page 3 states that a variable cannot reach its limit. The author follows Edwards's Differential Calculus and adopts Homersham Cox's proof of Taylor's theorem. That gentleman's first name, however, suffers somewhat at his hands. On page 147 a singular point is defined to have some peculiarity not depending on the position of the coördinate axes; we are then told that the most important singular points are maxima and minima.

*Elements of the Differential and Integral Calculus*, with examples and practical applications. By J. W. NICHOLSON, President and Professor of Mathematics in the Louisiana State University and Agricultural and Mechanical College. New York and New Orleans, University Publishing Company, 1896. 8vo, pp. xi+256.

This book, while of about the same size as the preceding one, is more suggestive, an attempt being made to introduce some of the more precise ideas of modern mathematics. The differential of a function is defined to be that part of its increment which varies proportionally with the increment of the independent variable. This definition is the one now recognized by most authorities as the most satisfactory. The author, however, denoting the derivative by $m_1$, writes $\Delta y = m_1 h + m_2 h^2$, a form which appears to involve the existence of the second derivative as well as the first. In fact, in a note ($A_1$) on the last page of the work he says that such a form implies not only existence but continuity of the second derivative. In another note ($A_3$) at the end of the work the author criticises the grounds assigned by Byerly and by Rice and Johnson for making $d(dx) = 0$. He contends that the differential of $dx$ is zero, because $dx$ as a variable is independent of $x$. This, of course, is not sound. If

a variable $y$ is independent of another variable $x$, it is true that we may still write

$$dy = \frac{dy}{dx}\,dx;$$

but the coefficient of $dx$ is not a partial derivative, and $dy$, therefore, instead of being zero, is indeterminate. In order that $d\,(dx)$ may be zero, we must assume that $dx$ takes the same value for all values of $x$. This assumption, however, does not prevent our varying $dx$ from one instant to another in a perfectly arbitrary manner.

*Elements of Differential Calculus.* By EDGAR W. BASS, Professor of Mathematics in the U. S. Military Academy. New York, John Wiley & Sons, 1896. 8vo, viii+354.

In regard to its logical development and its applications, both analytic and geometric, this text-book is one of the most complete yet published in America. The subject is introduced by a modified method of rates. Some of the author's definitions will sufficiently reveal his procedure.

Any variable which approaches zero as a limit is called an infinitesimal. The measure of the relative degree of rapidity of change of a function and its variable at any state, is called the rate of change of the function with respect to the variable for that state. An arbitrary amount of change assumed for the independent variable is called the differential of the variable. The differential of a function of a single variable is the change that the function would undergo from any state, were it to retain its rate of change at that state while the variable is changed by its differential. The total differential of a function of two variables is the change that the function would undergo from any state, were it to retain its rate at that state with respect to each variable while both variables changed by their differentials.

It follows that $\Delta x$ is an infinitesimal, but that $dx$ is not. This is contrary to the prevailing tendency which makes $\Delta x = dx$.* We notice two respects in which the author departs without sufficient warrant, we think, from the prevailing usage. He symbolizes the statement that $x$ approaches $a$ as a limit by placing between $x$ and $a$ an arrow pointing toward $a$. Further in writing partial derivatives,

* Byerly's Differential Calculus (1879) was, we believe, the first text-book in the English language to give this equation as an immediate consequence of the definition of a differential.

he uses the symbol $\partial$ as a prefix only for functions, retaining $d$ with the independent variables of the denominator.

*The Calculus for Engineers.* By JOHN PERRY, M.E., D.Sc., F.R.S., Professor of Mechanics and Mathematics in the Royal College of Science, London. London and New York, Edward Arnold, 1897. 8vo, pp. vii+378.

This very unconventional text-book is one of the most interesting and suggestive works that we have ever seen. It consists of three chapters, the first containing 160 pages, the second and third each one hundred pages. Chapters I. and II. are packed full of illustrations drawn from the practical problems of engineering. In the first few pages we find among the subjects discussed: mechanisms, speed, acceleration, kenetic energy, elongation of a spring, voltage of an electric circuit, and work of a steam engine. Maxima and minima follow with applications to strength of beams, economy of fuel, electric currents, indicator diagrams, electric conductors, electric traction, the suspension bridge, and heating surface of a boiler. Next under integration we find: moment of inertia, center of gravity, attraction, strength of thick and thin cylinders, gas engines, elasticity, friction, bending of beams, pressure in fluids, fluid motion, gases, liquids, magnetic field, self induction of two parallel wires, and the thermodynamical laws. Passing to the second chapter, we notice: leakage resistance of cables and condensers, Newton's law of cooling, slipping of a belt, atmospheric pressure, fly-wheel with fluid frictional resistance, electric conductor, coil of an alternator, bifilar suspension, connecting rod, valve gears, beats in music, tides, rotating magnetic field, alternating current power, mechanical vibration, forced vibrations, vibration indicator, natural vibrations, network of conductors, idle currents of transformers, more than a score of other electrical problems, and strength of struts.

A good idea of the plan and style of the book may be obtained from the introductory sentences of Chapter III. We quote: "In Chapter I. we dealt only with the differentiation and integration of $x^n$ and in Chapter II. with $e^{ax}$ and $\sin ax$, and unless one is really intending to make a rather complete study of the calculus, nothing further is needed. Our knowledge of those three functions is sufficient for nearly every practical engineering purpose. It will be found, indeed, that many of the examples given in this chapter might have been given in Chapters I. and II. For the differentiation and integration of functions in general,

we should have preferred to ask students to read the regular treatises, skipping difficult parts in a first reading and afterwards returning to these parts when there is the knowledge which it is necessary to have before one can understand them. If a student has no tutor to mark these difficult parts for him, he will find them out for himself by trial.

"*By means of a few rules* it is easy to become able to differentiate any algebraic functions of $x$, and, in spite of our wish that students should read the regular treatises, we are weak enough to give these rules here. They are mainly used to enable schoolboys to prepare for examinations and attain facility in differentiation. These boys so seldom learn more of this wonderful subject, and so rapidly lose the facility in question, because they never have learnt really what $dy/dx$ means, that we are apt with beginners to discourage much practice in differentiation, and so err, possibly, as much as the older teachers, but in another way. If, however, a man sees clearly the object of his work, he ought to try to gain this facility in differentiation and to retain it. The knack is easily learnt, and in working the examples he will, at all events, become more expert in manipulating algebraic and trigonometric expressions, and such expertness is all-important to the practical man.

"In Chapters I. and II. we thought it very important that students should graph several illustrations of

$$y = ax^n, \quad y = ae^{bx}, \quad y = a \sin (bx + c).$$

So also they ought to graph any new function which comes before them. But we would again warn them that it is better to have graphed a few very thoroughly, than to have a hazy belief that one has graphed a great number.

"The engineer discovers himself and his own powers in the first problem of any kind that he is allowed to work out completely by himself. The nature of the problem does not matter; what does matter is the thoroughness with which he works it out."

The author is sometimes quite forcible, for example: "Now surely there is no such great difficulty in catching the idea of a limiting value. Some people have the notion that we are stating something that is only approximately true; it is often because their teacher will say such things as 'reject 16.1 $\delta t$ because it is small,' or 'let $dt$ be an infinitely small amount of time' and they proceed to divide something by it, showing that although they may reach the age of Methuselah they will never have the common sense

of an engineer." For a specimen of just this kind of work, see the deduction of

$$\partial p = \left(\frac{dp}{dt}\right) \partial t + \left(\frac{dp}{dv}\right) \partial v$$

on page 138 of the book under review. Again, we read: "In some of the following integrals certain *substitutions* are suggested. The student must not be discouraged if he cannot see why these are suggested; these suggestions are the outcome of, perhaps, weeks of mental effort by some dead and gone mathematician. Indeed, some of them are no better than this, that we are told the answer and are merely asked to test if it is right by differentiation."

The book was constructed in part from lectures which the author delivered at a night school. In several places it contains traces of the excitement under which he labored in his anxiety to make his point. The intelligent reader, however, will have no difficulty in recognizing such places and in accepting his statements at their proper discount.

THOMAS S. FISKE.

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## ERRATA.

OUR attention has been called to several errata occurring in previous numbers of the BULLETIN. In Mr. Macaulay's article, "Newton's theory of kinetics," published in the number for July, 1897, of the last volume, the following corrections should be made:

| | | | | | | |
|---|---|---|---|---|---|---|
| p. 364, | line | 28, | *for* | in | *read* | is |
| " | " | 29, | " | is | " | as |
| " | " | 44, | " | equally | " | equably |
| p. 366, | " | 6, | " | equal | " | great. |

In Dr. McClintock's article "On a solution of the biquadratic which combines the methods of Descartes and Euler," contained in the same number, a correction should be made:

p. 389, line 24,

$$\text{for } 4rv = v(p + v^2) - q^2$$
$$\text{read } 4rv = v(p + v)^2 - q^2.$$

In the report by the Secretary "Fourth Summer Meeting of the American Mathematical Society," contained in the BULLETIN for October, 1897, a misprint occurs:

p. 8, line 41,
p. 9, lines 1, 2, 3, 8, } in the formulæ, *for e read* $\epsilon$.

## NOTES.

THE Spring meeting of the Chicago section of the AMERICAN MATHEMATICAL SOCIETY will take place on Saturday, April 9th, at the University of Chicago.

A NEW list of members of the AMERICAN MATHEMATICAL SOCIETY has just been published and distributed to the members of the Society. Extra copies may be obtained from the Secretary.

THE Lobachevsky prize for 1897 has been awarded to Professor SOPHUS LIE, of the University of Leipzig, for the third volume of his work "Theorie der Transformationsgruppen," Leipzig, 1893. The decision was announced at the meeting of the Physico-Mathematical Society of Kazan, November 3, 1897. The following works were awarded honorable mention: L. GÉRARD, Thèse sur la géométrie noneuclidienne, Paris, 1892; E. CESÀRO, Lezioni di geometria intrinseca, Napoli, 1896; G. FONTENÉ, L'hyperespace à $n-1$ dimensions, Paris, 1892. Nine works had been submitted to compete for the prize which amounts to 500 rubles. A gold medal was also awarded to Professor FELIX KLEIN, of the University of Göttingen, for the assistance he rendered the committee in assigning the prize. Professor Klein's report, Zur ersten Verteilung des Lobatschewsky-Preises, Gutachten betreffend den dritten Band der Theorie der Transformationsgruppen von S. Lie, has just been published. The next award of the Lobachevsky prize, which is given for a work on geometry, preferably noneuclidean geometry, published during the six years preceding the time of award, will take place on the third of November, 1900; all competing works must be sent to the Kazan Physico-Mathematical Society before November, 3, 1899.

AMONG the prizes awarded at the annual meeting of the French Academy of Sciences were the following: *Prix Francœur* (1,000 francs), awarded annually for discoveries or works useful to the progress of mathematical science, to G. ROBIN for the whole of his mathematical works; *Prix Poncelet* (2,000 francs), awarded annually for the work, published during the preceding ten years, which, in the judgment of the Academy, has been most useful to the progress of mathematical science, to R. LIOUVILLE, for the whole of his mathematical and mechanical works.

A statement of the subjects set by the Academy for the next mathematical prizes is given in the BULLETIN for March, 1897.

AT Christmas, 1897, an album containing the photographs of the present and former members of the Mathematischer Verein at Göttingen, was presented to Professor Felix Klein, in remembrance of the fact that his activity as professor has extended through a period of twenty-five years. A further collection for presentation to Professor Klein is being made of the photographs and signatures of all his former students at Göttingen.

THE International Society for the promotion of quaternions and allied mathematics has elected the following officers: President, Sir Robert S. Ball, Cambridge, England; General Secretary, Professor A. Macfarlane, Lehigh University, Bethlehem, Pa.

THE late Professor BRIOSCHI afforded a striking instance of a combination rare in this generation. Though a pure mathematician of the first rank, he had also a distinctly practical side. The Polytechnic School at Milan was founded through his efforts, and at the time of his death he was professor in this institution not only of pure mathematics but also of hydraulics. It appears from the catalogue that his course in the latter subject was concerned with matters of the most immediately practical interest, such as, for example, the damming of rivers.

WE have recently received several catalogues of rare and second-hand mathematical books. Specially worthy of mention are: Catalogue No. 58 of A. Hermann, 8 rue de la Sorbonne, Paris; Antiquariatskatalog Nr. 536 of K. F. Koehler's antiquarium, Kurprinzstrasse 6, Leipzig; and Circular No. 129 of William Wesley & Son, 28 Essex Street, Strand, London. That last mentioned includes many works from the library of the Rev. A. Freeman, late Fellow of St. John's College, Cambridge, England.

PROFESSOR FRIEDRICH WINNECKE, distinguished astronomer, died at Bonn, December 3, 1897.

L. CREMONA, professor of higher mathematics in the University of Rome, has been chosen to succeed the late Professor Sylvester as corresponding member of the Paris Academy of Sciences.

PROFESSOR CHARLOTTE ANGAS SCOTT, of Bryn Mawr College, Bryn Mawr, Pennsylvania, has recently been appointed a collaborator of the "Revue Semestrielle des Publications Mathématiques," with special reference to such American mathematical periodicals as are not already in charge. It is particularly requested that editors whose periodicals have not hitherto been analyzed in the *Revue* will kindly communicate with her without delay.

# NEW PUBLICATIONS.

## I. HIGHER MATHEMATICS.

BERNHARD. Ueber lineare Schaaren von Kurven und Flächen. Ehingen, 1897. 4to. 30 pp. Mk. 1.80

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